$$V=4\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\phi}\int_{a\sin\phi\sec\theta}^{\frac{1}{2}\pi}\rho^2\sin\phi d\phi d\theta d\rho=\tfrac{4}{3}a^3\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\phi}(1-\sin^3\phi\sec^3\theta)\sin\phi d\phi d\theta,$$

$$=\tfrac{4}{3}a^3[\tfrac{1}{2}\pi-1\tfrac{5}{6}+\tfrac{3}{16}\int_0^{\frac{1}{2}\pi}\log\tan\frac{\phi}{2}d\phi=\tfrac{2}{3}\pi a^3-\tfrac{5}{4}a^3+\tfrac{1}{4}a^3\int_0^{\frac{1}{2}\pi}\log\tan\frac{\phi}{2}d\phi],$$

$$=\tfrac{2}{3}\pi a^3-\tfrac{5}{4}a^3-\tfrac{1}{4}a^3\sum_{n=1}^{n=\infty}\frac{2n-1}{[4n-3]^2[4n-1]^2}=.386a^3,\text{ nearly.}$$

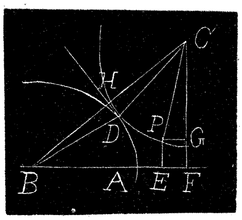
On the integration of $\int_0^{\frac{1}{2}\pi}\log\tan\frac{\phi}{2}d\phi$, see the remarks on Prize Problem, No. 123, Calculus.

157. Proposed by **L. C. WALKER, A. M.,** Graduate Student, Leland Stanford Jr. University, Cal.

Two equal ellipses are tangent to each other at the vertices of the major axes. If one of them be rolled on the other; find (1) the equation and area of the curve described by the vertex, and (2) by the center.

Solution by **G. B. M. ZERR, A.M., Ph.D.,** Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and the **PROPOSER.**

Let a and b be the semi-axes of both ellipses; B the center of the fixed ellipse: C, the center of the rolling ellipse; P, its vertex; and D, the point of contact; $e^2=(a^2-b^2)/a^2$. Let $BE=x$, $BF=m$, $PE=y$, $CF=n$, and $\angle ABC=\angle PCB=\theta$.



Then, $BC=2a\sqrt{(1-e^2\sin^2\theta)}$=twice the length of the perpendicular from B on the tangent at D.

$m=2a\cos\theta\sqrt{(1-e^2\sin^2\theta)}$, $n=2a\sin\theta\sqrt{(1-e^2\sin^2\theta)}$,

$x=m-PG=2a\cos\theta\sqrt{(1-e^2\sin^2\theta)}-a\cos2\theta$,

$y=n-CG=2a\sin\theta\sqrt{(1-e^2\sin^2\theta)}-a\sin2\theta$.

$x^2+y^2=r^2=a^2+4a^2(1-e^2\sin^2\theta)-4a^2\cos\theta\sqrt{(1-e^2\sin^2\theta)}$,

the equation of the locus of the vertex.

$$\text{Area}=a^2\int_0^{\pi}[5-4e^2\sin^2\theta-4\cos\theta\sqrt{(1-e^2\sin^2\theta)}]d\theta=\pi a^2(5-2e^2)=\pi(3a^2+2b^2).$$

$BC^2=\rho^2=4a^2(1-e^2\sin^2\theta)$, the equation of the locus of the center.

$$\text{Area}=4a^2\int_0^{\pi}(1-e^2\sin^2\theta)d\theta=2\pi a^2(2-e^2)=2\pi(a^2+b^2).$$

Also solved by *J. SCHEFFER*, and *G. W. GREENWOOD*.

MECHANICS.

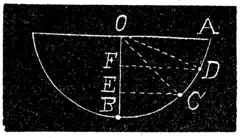
147. Proposed by **W. J. GREENSTREET, M. A.,** Editor of The Mathematical Gazette, Stroud, England.

A particle mass m is attached to one end of a string, the other end of which is fixed. It is projected horizontally with such a velocity that it would rise to a position in which

the string would be horizontal. But on its upward path it meets an inelastic particle mass m' and the height to which it rises is diminished by $1/p$th of what it would have risen. Find m', and the tensions of the string just after collision and at the greatest height of the particle.

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let B be the point of projection of particle mass m; C, the position of m' supposed at rest; D, the point at which m stops after impact with m'. Let $\angle COB=\theta$, $BO=a$.. Then $OF=a/p$, $FB=a(p-1)/p$, $OE=a\cos\theta$. Velocity of projection $=\sqrt{(2ga)}$, velocity of m at C at moment of impact $=\sqrt{(2ga\cos\theta)}$.



$\therefore m\sqrt{(2ga\cos\theta)}=v'(m+m')$.

But $v'=\sqrt{(2g.FE)}=\sqrt{[(2ga/p)(p\cos\theta-1)}$.

$\therefore m\sqrt{(2ga\cos\theta)}=(m+m')\sqrt{[(2ga/p)(p\cos\theta-1)}$.

$$\therefore m'=\frac{m\{\sqrt{[\cos\theta]}-\sqrt{[(p\cos\theta-1)/p]}\}}{\sqrt{[(p\cos\theta-1)/p]}}.$$

Let $(m+m')g=W$. The tension is composed of the tension due to acceleration imparted by the central force plus the tension due to gravity. Let $T=$ tension just after impact, $t=$tension at highest point.

Then $T=\dfrac{Wv'^2}{ga}+W\cos\theta=(2W/p)(p\cos\theta-1)+W\cos\theta=3W\cos\theta-2W/p$.

Since velocity is zero at D, $t=W\cos DOB=W/p$. $\therefore t=W/p$.

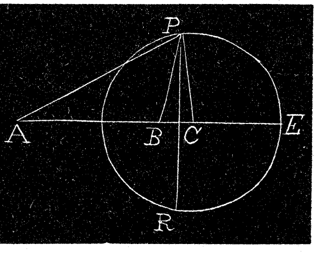
As m' is inelastic, it is supposed that m' coalesces with m.

148. Proposed by G. H. HARVILL, A. M., Malakoff, Texas.

Show that a law of density for points in space may be assumed such that the joint mass of any two points which are *electrical images* of each other in respect to a given sphere may be constant, and that their centers of gravity should lie on the surface of the sphere.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let A, B be the electrical images, a =radius of sphere, C its center, $AC=x$, $BC=y$. Let e=charge of electricity situated at A; and let the sphere be insulated and have a charge $-(ea^3/x^3)$. Let PR be the line of no electrification. Let m=mass of each point on surface of sphere; u =mass of point at A; v=mass of point at B; $\angle BAP=\theta$. Now $AP=AC$, $BP=PC$ (since PR is line of no electrification). Also



$$u+v=2m....(1),$$
$$u(x-a)=v(a-y)....(2).$$

From triangles ACP and ABP we have, since $AP=AC=x$, $BP=PC=a$,

$$\cos\theta=\frac{2x^2-a^2}{2x^2}=\frac{x^2+(x-y)^2-a^2}{2x(x-y)}.$$